JOURNAL OF COMPLEXITY **14**, 210–233 (1998)
ARTICLE NO. CM980474

Lyapunov Exponents versus Expansivity and Sensitivity in Cellular Automata*

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Received November 11, 1996

We establish a connection between the theory of Lyapunov exponents and the properties of expansivity and sensitivity to initial conditions for a particular class of discrete time dynamical systems; cellular automata (CA). The main contribution of this paper is the proof that all expansive cellular automata have positive Lyapunov exponents for almost all the phase space configurations. In addition, we provide an elementary proof of the non-existence of expansive CA in any dimension greater than 1. In the second part of this paper we prove that expansivity in dimension greater than 1 can be recovered by restricting the phase space to a *suitable* subset of the whole space. To this extent we describe a 2-dimensional CA which is expansive over a *dense uncountable* subset of the whole phase space. Finally, we highlight the different behavior of expansive and sensitive CA for what concerns the speed at which perturbations propagate. © 1998 Academic Press

* A preliminary version of this paper appeared in the Proceedings of the 2nd Conference on Cellular Automata for Research and Industry (ACRI'96), Milan, Italy, 1996.

0885-064X/98 $25.00

1. INTRODUCTION

The notion of chaos is very appealing, and it has intrigued many scientists (see [2, 3, 14, 17, 20] for some works on the properties that characterize a chaotic process). In the case of discrete time dynamical systems (DTDS) defined on a metric space, many definitions of chaos are based on the notion of sensitivity (see for example [8, 13, 17]). We now recall the definition of sensitivity to initial conditions for a generic DTDS (X, F). Here, we assume that X is equipped with a distance d and that the map $F: X \to X$ is continuous on X according to the metric topology induced by d.

Definition 1 (Sensitivity). A DTDS (X, F) is sensitive to initial conditions if and only if there exists $\delta > 0$ such that

$$\forall x \in X \quad \forall \varepsilon > 0 \quad \exists y \in X \quad \exists n \geqslant 0: \quad d(x, y) < \varepsilon \quad \text{and} \quad d(F^n(x), F^n(y)) > \delta.$$

The value δ is called the sensitivity constant.

Intuitively, a map is sensitive to initial conditions, or simply sensitive, if there exist points arbitrarily close to x which eventually separate from x by at least δ under iteration of F. We emphasize that not all points near x need eventually separate from x, but there must be at least one such point in every neighborhood of x. If a map is sensitive to initial conditions, then, for all practical purposes, the dynamics of the map defies numerical approximation. Small errors in computation which are introduced by round-off may become magnified upon iteration. The results of numerical computation of an orbit, no matter how accurate, may be completely different from the real orbit.

A property stronger than sensitivity is expansivity. Expansivity differs from sensitivity in that all nearby points eventually separate by at least δ. It is easy to verify that expansive CA are sensitive to initial conditions.

Definition 2 (Expansivity). A DTDS (X, F) is expansive if and only if there exists $\delta > 0$ such that

$$\forall x, y \in X \quad x \neq y \quad \exists n \geqslant 0: \qquad d(F^n(x), F^n(y)) > \delta.$$

The value δ is called the expansivity constant.

Sometimes the definition of expansivity given above is referred to as *forward* or *positive* expansivity in order to distinguish it from the notion of expansivity given for *invertible* (one-to-one) dynamical systems where "$\exists n \geqslant 0$" is replaced by "$\exists n \in \mathbf{Z}$."

In the case of differentiable spaces there is another parameter which is often used for detecting chaotic behaviors: Lyapunov exponents. We define them for a map $F\colon I \to I$, where I is a real interval.

DEFINITION 3 (Lyapunov Exponents). Let (I, F) be a DTDS. The Lyapunov exponent $\lambda(x)$ of $x \in X$ is defined by

$$\lambda(x) = \lim_{n \to \infty} \frac{1}{n} \log \left(\frac{dF^n(x)}{dx} \right).$$

Lyapunov exponents can be easily generalized to higher dimensions. Usually, a DTDS (I, F) is said to be chaotic at $x \in X$ if and only if $\lambda(x) > 0$.

In this paper we wish to discuss the role played by *time* in the definitions of expansivity, sensitivity and Lyapunov exponents. According to Definition 2, a DTDS (X, F) is said to be expansive if and only if after an *unspecified* number of applications of F, every pair of configurations, no matter how close they are, are separated by a preassigned constant value δ. The same consideration can be done for a sensitive DTDS. (X, F) is expansive even if the number of iterations needed for separating is of the order of 10^{100}. In other words, a quantitative measure of time does not come into play in determining the expansivity of a DTDS. This is one of the main criticisms made by those who prefer a Lyapunov exponent based approach for defining chaos.

Time plays a fundamental role in the definition of Lyapunov exponents. In fact, if $\lambda(x) > 0$, we have

$$\frac{dF^n(x)}{dx} \simeq \alpha^{n\lambda(x)},$$

where $\alpha > 1$. This means that F at x shows an exponential rate divergence in time. Unfortunately, Lyapunov exponents have many other drawbacks. The main one is that in many cases they cannot be computed in a close form and one needs to approximate them by time consuming and, sometimes, unreliable computer simulations (as in the case of CA).

In this paper we prove that, for cellular automata, the criticisms made by the supporters of the Lyapunov exponents to the expansivity property are not well founded. In fact, we show that every expansive CA must have almost all Lyapunov exponents uniformly bounded away from zero by a constant δ which only depends on the CA we consider. In other words, expansivity implies positive Lyapunov exponents. Note that the task of verifying expansivity appears to be simpler than the computation of the Lyapunov exponents. In [9], the authors define a large class of expansive CA which contains additive and non-additive ones, while in [16] the class of expansive additive CA is characterized in terms of a simple property of the coefficients of the local rule.

The main results of this paper can be summarized as follows.

1. Let (X, F) be any expansive CA with expansivity constant δ. Let $x, y \in X$ be any pair of distinct configurations whose distance is ε. The number of iterations needed by F for separating x and y by at least δ depends only on ε and is of the order of $\log(\delta/\varepsilon)$ (Corollary 4.5). In addition, we show that a similar result *does not* hold for sensitive CA (Example 3).

2. Every expansive CA (X, F) has positive (uniformly bounded away from zero) Lyapunov exponents over a set $Y \subseteq X$ of configurations of full measure. A slightly weaker result holds also for those configurations belonging to $X \setminus Y$ (Theorem 5.2).

3. We provide an elementary proof of the non existence of D-dimensional CA for $D \geqslant 2$ (Theorem 4.4). A (much more complex) proof of this result has been given by Shereshevsky [19, Corollary 2] in the more general setting of group actions by endomorphisms. We also show that expansivity can be achieved in dimension greater than 1 if we restrict ourselves to a suitable subset of the phase space. More precisely, we show that there exists a 2-dimensional CA (X, F) which is expansive on a dense invariant subset of X (Theorem 6.3).

The rest of this paper is organized as follows. In Section 2 we give basic definitions and notations. In Section 3 we review the notion of Lyapunov exponents for CA and we recall some known results concerning CA and Lyapunov exponents. In Section 4 we prove the main results on expansive CA and we highlight some differences among expansive CA and general expansive maps. In Section 5 we show the relationship between expansivity and Lyapunov exponents in CA. In Section 6 we prove the existence of CA which are expansive over a dense subset of the whole space but are not globally expansive. In Section 7 we compare the behavior of expansive and sensitive CA for what concerns the number of iterations required to separate neighboring configurations. Section 8 contains some concluding remarks.

2. DEFINITIONS AND NOTATIONS

Let $\mathscr{A} = \{0, 1, ..., p-1\}$ be a finite alphabet of cardinality $p \geqslant 2$. We consider the *space of configurations*

$$\mathscr{A}^{\mathbf{Z}^D} = \{c \mid c\colon \mathbf{Z}^D \to \mathscr{A}\},$$

which consists of all functions from $\mathbf{Z}^D$ into $\mathscr{A}$. For example, each element of $\mathscr{A}^{\mathbf{Z}^2}$ can be visualized as an infinite 2-dimensional lattice in which each cell contains an element of $\mathscr{A}$.

Let $s \geqslant 1$. A *neighborhood frame* of size s is an ordered set of distinct vector $\vec{u}_1, \vec{u}_2, ..., \vec{u}_s \in \mathbf{Z}^D$. Given $f: \mathscr{A}^s \to \mathscr{A}$, a D-dimensional CA based on the *local rule* f is the pair $(\mathscr{A}^{\mathbf{Z}^D}, F)$, where $F: \mathscr{A}^{\mathbf{Z}^D} \to \mathscr{A}^{\mathbf{Z}^D}$, is the *global transition map* defined as follows. For every $c \in \mathscr{A}^{\mathbf{Z}^D}$ the configuration $F(c)$ is such that for every $\vec{v} \in \mathbf{Z}^D$,

$$[F(c)](\vec{v}) = f(c(\vec{v}+\vec{u}_1), ..., c(\vec{v}+\vec{u}_s)).$$

In other words, the content of cell $\vec{v}$ in the configuration $F(c)$ is a function of the content of cells $\vec{v}+\vec{u}_1, ..., \vec{v}+\vec{u}_s$ in the configuration c. Note that the local rule f and the neighborhood frame completely determine F.

EXAMPLE 1. When $s=1$ and f is the identity map, we have

$$[F(c)](\vec{v}) = c(\vec{v}+\vec{u}_1).$$

Hence, the global transition map F is simply a shift of the configuration space $\mathscr{A}^{\mathbf{Z}^D}$. In this case, we say that F is a *shift map* and we denote it by $\sigma^{\vec{u}_1}$. A fundamental property of the shift maps is that they commute with the global map of any other CA. That is, for any $\vec{u} \in \mathbf{Z}^D$ and $c \in \mathscr{A}^{\mathbf{Z}^D}$, we have $\sigma^{\vec{u}}(F(c)) = F(\sigma^{\vec{u}}(c))$.

For 1-dimensional CA, we use a simplified notation. Let $f: \mathscr{A}^{2k+1} \to \mathscr{A}$, be any map. A 1-dimensional CA based on the local rule f is a pair $(\mathscr{A}^{\mathbf{Z}}, F)$, where $F: \mathscr{A}^{\mathbf{Z}} \to \mathscr{A}^{\mathbf{Z}}$, is defined by

$$[F(c)](i) = f(c(i-k), ..., c(i+k)), \qquad c \in \mathscr{A}^{\mathbf{Z}}, \quad i \in \mathbf{Z}.$$

We say that k is the *radius* of f. Note that, even if $f(x_{-k}, ..., x_{-1}, x_0, x_1, ..., x_k)$ must depend on at least one between x_{-k} and x_k, in general f does not depend on all the $2k+1$ variables $x_{-k}, ..., x_k$.

EXAMPLE 2. An important 1-dimensional CA is the *right shift map* $(\mathscr{A}^{\mathbf{Z}}, \sigma)$ defined by

$$[\sigma(c)](i) = c(i-1).$$

The global map σ corresponds to the local rule $f(x_{-1}, x_0, x_1) = x_{-1}$. The inverse of σ is the *left shift map* defined by $[\sigma^{-1}(c)](i) = c(i+1)$ which corresponds to the local rule $f(x_{-1}, x_0, x_1) = x_1$. For $j \geqslant 0$ the iterated map σ^j is such that

$$[\sigma^j(c)](i) = c(i-j), \qquad c \in \mathscr{A}^{\mathbf{Z}}, \quad i \in \mathbf{Z}. \tag{1}$$

In the following we use σ^j with $j<0$ to denote the map σ^{-1} iterated $|j|$ times. Note that, using this notation, (1) holds for any $j \in \mathbf{Z}$.

In order to specialize the notions of sensitivity and expansitivity to the case of D-dimensional CA, we need to introduce a distance mapping over the space $\mathscr{A}^{\mathbf{Z}^D}$. In the literature there are many examples of distance mappings over $\mathscr{A}^{\mathbf{Z}^D}$ (see for example [4, 5, 10, 11, 15]). Most of them induce over $\mathscr{A}^{\mathbf{Z}^D}$ the so-called *product* topology.[1] With this topology, $\mathscr{A}^{\mathbf{Z}^D}$ is a compact and totally disconnected space and every CA is a uniformly continuous map.

Among all the distances over $\mathscr{A}^{\mathbf{Z}^D}$ that induce the product topology we use the following one which enables us to prove our results in the simplest way. Given $x, y \in \mathscr{A}^{\mathbf{Z}^D}$ such that $x \neq y$ we define

$$\langle x, y \rangle_\infty = \min\{ \|\vec{v}\|_\infty \mid \vec{v} \in \mathbf{Z}^D \text{ and } x(\vec{v}) \neq y(\vec{v})\},$$

where $\|\vec{v}\|_\infty$ is the maximum of the absolute value of the components of $\vec{v}$. Define

$$d(x, y) = \begin{cases} 0, & \text{if } \ x = y, \\ 2^{-\langle x, y \rangle_\infty}, & \text{if } \ x \neq y. \end{cases} \tag{2}$$

The distance d has been used for example in [4, 15].

Throughout the paper, $F(c)$ will denote the result of the application of the map F to the configuration c, and $c(\vec{v})$ will denote the value of the entry with coordinates $\vec{v}$ of the configuration c. We recursively define $F^n(c)$ by $F^n(c) = F(F^{n-1}(c))$, where $F^0(c) = c$.

3. LYAPUNOV EXPONENTS FOR CA

The notion of Lyapunov exponents given in Definition 3 can be applied only to differentiable spaces. Since $\mathscr{A}^{\mathbf{Z}}$ is not a differentiable space, for CA we need an *ad hoc* definition. In this section we recall the definition of

[1] The product topology over $\mathscr{A}^{\mathbf{Z}^D}$ is that induced by the discrete topology on $\mathscr{A}$.

Lyapunov exponents for the special case of 1-dimensional CA given in [18]. There, the authors introduce quantities analogous to Lyapunov exponents of smooth dynamical systems with the aim of describing the *local instability* of orbits in CA.

For every $x \in \mathscr{A}^{\mathbf{Z}}$ and $s \geqslant 0$ we set

$$W_s^-(x) = \{ y \in \mathscr{A}^{\mathbf{Z}} \colon y(i) = x(i) \text{ for all } i \leqslant -s \},$$

$$W_s^+(x) = \{ y \in \mathscr{A}^{\mathbf{Z}} \colon y(i) = x(i) \text{ for all } i \geqslant s \}.$$

We have that $W_i^+(x) \subset W_{i+1}^+(x)$ and $W_i^-(x) \subset W_{i+1}^-(x)$. For every $n \geqslant 0$ we define

$$\tilde{\Lambda}_n^-(x) = \min\{ s \geqslant 0 \colon F^n(W_0^-(x)) \subset W_s^-(F^n(x)) \},$$

$$\tilde{\Lambda}_n^+(x) = \min\{ s \geqslant 0 \colon F^n(W_0^+(x)) \subset W_s^+(F^n(x)) \}.$$

Intuitively, for the CA defined by F the value $\tilde{\Lambda}_n^+(x)$ $[\tilde{\Lambda}_n^-(x)]$ measures how far a perturbation front moves right [left] in time n if the front is initially located at $i = 0$. Finally, we consider the shift invariant quantities

$$\Lambda_n^-(x) = \max_{j \in \mathbf{Z}} \tilde{\Lambda}_n^-(\sigma^j(x)), \qquad \Lambda_n^+(x) = \max_{j \in \mathbf{Z}} \tilde{\Lambda}_n^+(\sigma^j(x)),$$

where σ denotes the right shift map defined in Example 2. Intuitively, the value $\tilde{\Lambda}_n^+(\sigma^j(x))$ $[\tilde{\Lambda}_n^-(\sigma^j(n))]$ measures how far a perturbation front moves right [left] in time n if the front is initially located at j.

The values $\lambda^+(x)$ and $\lambda^-(x)$ defined by

$$\lambda^+(x) = \lim_{n \to \infty} \frac{1}{n} \Lambda_n^+(x) \qquad \lambda^-(x) = \lim_{n \to \infty} \frac{1}{n} \Lambda_n^-(x) \tag{3}$$

are called respectively the right and left Lyapunov exponents of the CA F for the configuration x. The limits in (3) do not necessarily exist for all $x \in \mathscr{A}^{\mathbf{Z}}$. However, the following result holds.

THEOREM 3.1 [18]. *For any σ-invariant and F-invariant measure μ defined on $\mathscr{A}^{\mathbf{Z}}$, there exists a set $Y \subseteq X$ of full measure ($\mu(Y) = 1$) such that for every $x \in Y$ the limits* (3) *exist.*

Typical examples of σ-invariant and F-invariant measures are the so-called *Bernoulli product measures* (see [7] for details).

4. SOME PROPERTIES OF EXPANSIVE CELLULAR AUTOMATA

In this section we study the properties of expansive functions over a compact metric space. In particular, we consider the case in which we are given a function $F: X \to X$ such that

$$\exists \lambda > 0: \qquad \forall x, y \in X, \quad d(F(x), F(y)) \leqslant \lambda d(x, y). \tag{4}$$

Using calculus terminology, if (4) holds we say that F is a Lipschitz function with parameter λ. The reason for which we are interested in Lipschitz functions is that the global transition map F associated to a CA always satisfies (4). In this case, the parameter λ can be easily obtained from the radius of the local rule.

For any pair $x, y \in X$, by (4) we have $d(F^n(x), F^n(y)) \leqslant \lambda^n d(x, y)$. hence, the distance $d(F^n(x), F^n(y))$ cannot grow arbitrarily fast. Our main interest is to get lower bounds on how fast this distance can grow for expansive maps. Our main purpose is to prove some properties of expansive CA which will be used in the following sections. However, in doing so we will also highlight the different behaviors of expansive CA with respect to general expansive Lipschitz functions.

Lemma 4.1. *Let (X, d) be a compact metric space, and $F: X \to X$ be an expansive Lipschitz function. Then, we can find $\varepsilon' > 0$ such that for all ε, $0 < \varepsilon < \varepsilon'$, there exists $n = n(\varepsilon)$ such that*

$$\forall x, y \in X \qquad \varepsilon \leqslant d(x, y) \leqslant \varepsilon' \;\Rightarrow\; \exists k \leqslant n: d(F^k(x), F^k(y)) > 2d(x, y). \tag{5}$$

Proof. Assume F is expansive with parameter δ, and let λ denote the Lipschitz constant of the function F. Note that we must have $\lambda > 1$ otherwise F cannot be expansive. We prove the theorem for $\varepsilon' = \delta/6$. Let $\varepsilon < \varepsilon'$. Assume by contradiction that

$$\forall n \quad \exists x_n, y_n: \qquad \varepsilon \leqslant d(x_n, y_n) \leqslant \varepsilon' \quad \text{and}$$
$$d(F^k(x_n), F^k(y_n)) \leqslant 2d(x_n, y_n) \quad \forall k \leqslant n.$$

Since X is a compact space, we can build two sequences x_i, y_i such that

$$\varepsilon \leqslant d(x_i, y_i) \leqslant \varepsilon', \qquad \lim_{i \to \infty} x_i = \tilde{x}, \qquad \lim_{i \to \infty} y_i = \tilde{y}, \tag{6}$$

and

$$d(F^k(x_i), F^k(y_i)) \leqslant 2d(x_i, y_i) \qquad \forall k \leqslant i. \tag{7}$$

Moreover, we can assume that

$$d(x_i, \tilde{x}) < \frac{\delta}{3}\lambda^{-i}, \qquad d(y_i, \tilde{y}) < \frac{\delta}{3}\lambda^{-i}. \tag{8}$$

By the triangle inequality, we have for all i,

$$d(\tilde{x}, \tilde{y}) \geqslant d(x_i, y_i) - d(\tilde{x}, x_i) - d(y_i, \tilde{y}) \geqslant \varepsilon - \frac{2\delta}{3}\lambda^{-i},$$

which implies $\tilde{x} \neq \tilde{y}$. For the expansivity of F there exists m such that $d(F^m(\tilde{x}), F^m(\tilde{y})) > \delta$. Using again the triangle inequality, together with (8), we get

$$\begin{aligned} &d(F^m(x_m), F^m(y_m)) \\ &\qquad \geqslant d(F^m(\tilde{x}), F^m(\tilde{y})) - d(F^m(\tilde{x}), F^m(x_m)) - d(F^m(\tilde{y}), F^m(y_m)) \\ &\qquad > \delta - \lambda^m d(\tilde{x}, x_m) - \lambda^m d(\tilde{y}, y_m) \\ &\qquad > \delta - \frac{\delta}{3} - \frac{\delta}{3} = \frac{\delta}{3}. \end{aligned}$$

Since $\varepsilon' \leqslant \delta/6$, we have

$$d(F^m(x_m), F^m(y_m)) > 2\varepsilon' \geqslant 2d(x_m, y_m),$$

which is impossible since it contradicts the hypothesis (7). ∎

Note that the above lemma can be applied several times to prove that we can have an arbitrarily large growth of the initial distance. More precisely, let ε' and $n(\varepsilon)$ be given as in Lemma 4.1. Then, for any integer $t > 0$ and $\varepsilon \leqslant \varepsilon' 2^{-t}$, we have

$$\varepsilon \leqslant d(x, y) \leqslant \frac{\varepsilon'}{2^t} \;\Rightarrow\; \exists k \leqslant (t+1)\, n(\varepsilon): d(F^k(x), F^k(y)) > 2^{t+1} d(x, y). \tag{9}$$

It is straightforward to verify that any map which satisfies (5) is expansive with parameter ε'. Next theorem establishes that it suffices that (5) holds on a dense subset of X to guarantee that F is expansive over the whole space.

THEOREM 4.2. *let* (X, d) *be a compact metric space, and* $F: X \to X$ *be a Lipschitz function. Let* Y *be any dense subset of* X *such that* $F(Y) \subseteq Y$. *If there exists* ε' *such that for all* $\varepsilon < \varepsilon'$ *we can find* $n = n(\varepsilon)$ *such that*

$$\forall x, y \in Y, \qquad \varepsilon \leqslant d(x, y) \leqslant \varepsilon' \;\Rightarrow\; \exists k \leqslant n: d(F^k(x), F^k(y)) \geqslant 2d(x, y), \tag{10}$$

then F *is expansive over* X.

Proof. We prove that F is expansive with parameter $\varepsilon'/3$ by showing that

$$\forall x, y \in X, \qquad x \neq y, \quad d(x, y) \leqslant \varepsilon'/4 \;\Rightarrow\; \exists k: d(F^k(x), F^k(y)) > \varepsilon'/3.$$

Let $x, y \in X$ with $d(x, y) \leqslant \varepsilon'/4$, and let $n = n(d(x, y)/2)$. Since Y is a dense subset, we can find $\tilde{x}, \tilde{y} \in Y$ such that

$$d(x, \tilde{x}) \leqslant d(x, y)/4, \qquad d(y, \tilde{y}) \leqslant d(x, y)/4, \tag{11}$$

and

$$\lambda^{n\lceil \log_2(2\varepsilon'/d(x, y))\rceil}\, d(x, \tilde{x}) < \varepsilon'/3, \qquad \lambda^{n\lceil \log_2(2\varepsilon'/d(x, y))\rceil}\, d(y, \tilde{y}) < \varepsilon'/3, \tag{12}$$

where λ denotes the Lipschitz constant for F (note that we must have $\lambda > 1$ otherwise (10) cannot hold). By (11), using the triangle inequality, we get

$$d(\tilde{x}, \tilde{y}) \geqslant d(x, y) - d(x, \tilde{x}) - d(y, \tilde{y}) \geqslant d(x, y)/2.$$

Since $\tilde{x}, \tilde{y} \in Y$, by (10) we know that there exists $k \leqslant n\lceil \log_2(\varepsilon'/d(\tilde{x}, \tilde{y}))\rceil$ such that

$$d(F^k(\tilde{x}), F^k(\tilde{y})) \geqslant \varepsilon'.$$

Moreover, since $d(\tilde{x}, \tilde{y}) \geqslant d(x, y)/2$, we have

$$k \leqslant n\lceil \log_2(2\varepsilon'/d(x, y))\rceil. \tag{13}$$

Using the triangle inequality, we get

$$\begin{aligned} d(F^k(x), F^k(y)) & \\ &\geqslant d(F^k(\tilde{x}), F^k(\tilde{y})) - d(F^k(x), F^k(\tilde{x})) - d(F^k(y), F^k(\tilde{y})) \\ &\geqslant \varepsilon' - \lambda^k d(x, \tilde{x}) - \lambda^k d(y, \tilde{y}) \\ &> \varepsilon' - \frac{\varepsilon'}{3} - \frac{\varepsilon'}{3} = \frac{\varepsilon'}{3}. \end{aligned}$$

where the last inequality follows from (13) and (12). ∎

We now show that Lemma 4.1 implies that for expansive CA any difference between two configurations propagates with a constant speed. In other words, if we get x' by modifying a configuration x, the iteration of an expansive map F will spread this "perturbation" with a speed which can be bounded from below. In addition, we give a bound which is uniform, that is, it holds for any configuration x. As we will see, this is a very strong characterization of expansive CA and we will use it many times in the rest of the paper.

LEMMA 4.3. *Let $(\mathscr{A}^{\mathbf{Z}^D}, F)$ be a D-dimensional CA over the finite alphabet $\mathscr{A}$. The map F is expansive if and only if there exist τ, m such that for all $x, y \in \mathscr{A}^{\mathbf{Z}^D}$, $x \neq y$, we have*

$$\langle x, y \rangle_\infty \geqslant \tau \quad \Rightarrow \quad \exists k \leqslant m \text{ such that } \langle F^k(x), F^k(y) \rangle_\infty < \langle x, y \rangle_\infty. \tag{14}$$

Proof. We prove the result for $D = 2$, the general case being analogous. The "only if" part is straightforward. If (14) holds, then for every $x, y \in \mathscr{A}^{\mathbf{Z}^2}$, $x \neq y$, there exists $n \geqslant 0$ such that $\langle F^n(x), F^n(y) \rangle_\infty < \tau$ which implies $d(F^n(x), F^n(y)) \geqslant 2^{-\tau}$. Hence, F is expansive with constant δ, for any $\delta < 2^{-\tau}$.

Assume now F is expansive. To prove the "if" part, we first show that (14) holds for all $x, y \in \mathscr{A}^{\mathbf{Z}^2}$ such that $\langle x, y \rangle_\infty = \tau$. Then, using the fact that F commutes with any 2-dimensional shift $\sigma^{(i,j)}$, we prove (14) also for $\langle x, y \rangle_\infty > \tau$. Let ε' be defined as in Lemma 4.1, and let τ be the smallest integer such that $2^{-\tau} \leqslant \varepsilon'$. By Lemma 4.1, we know that there exists m such that

$$2^{-\tau} \leqslant d(x, y) \leqslant \varepsilon' \quad \Rightarrow \quad \exists k \leqslant m\colon d(F^k(x), F^k(y)) > 2d(x, y). \tag{15}$$

Assume now $\langle x, y \rangle_\infty = \tau$. Since $d(x, y) = 2^{-\tau}$, by (15) there exists $k \leqslant m$ such that

$$d(F^k(x), F^k(y)) > 2d(x, y) = 2^{1-\tau},$$

which implies $\langle F^k(x), F^k(y) \rangle_\infty < \tau$ as claimed.

Consider now any $t > \tau$ and let $\langle x, y \rangle_\infty = t$. One can see that there exists $\vec{u} \in \mathbf{Z}^2$ such that $\|\vec{u}\|_\infty = t - \tau$ and $\langle \sigma^{\vec{u}}(x), \sigma^{\vec{u}}(y) \rangle_\infty = \tau$ (see Fig. 1). Since $\langle \sigma^{\vec{u}}(x), \sigma^{\vec{u}}(y) \rangle_\infty = \tau$, we know that there exists $k \leqslant m$ such that

$$[F^k(\sigma^{\vec{u}}(x))](\vec{v}) \neq [F^k(\sigma^{\vec{u}}(y))](\vec{v})$$

for some $\vec{v} \in \mathbf{Z}^2$ with $\|\vec{v}\|_\infty < \tau$. Since F^k and $\sigma^{\vec{u}}$ commutes, we get

$$[\sigma^{\vec{u}}(F^k(x))](\vec{v}) \neq [\sigma^{\vec{u}}(F^k(x))](\vec{u}),$$

which implies

$$[F^k(x)](\vec{v} + \vec{u}) \neq [F^k(y)](\vec{v} + \vec{u}).$$

This yields

$$\langle F^k(x), F^k(y) \rangle_\infty \leqslant \|\vec{v} + \vec{u}\|_\infty \leqslant \|\vec{v}\|_\infty + \|\vec{u}\|_\infty < \tau + (t - \tau) = t.$$

Hence, within m steps we have $\langle F^k(x), F^k(y) \rangle_\infty < \langle x, y \rangle_\infty$ as claimed. ∎

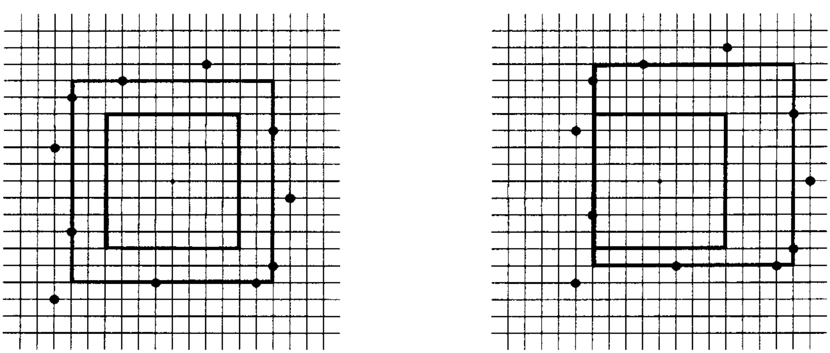

FIG. 1. Each lattice represents a portion of $\mathbf{Z}^2$ centered at the origin (0, 0). Black circles denote the sites where configurations differ. We have $\langle x, y \rangle_\infty = \tau + 2$ (left), and $\langle \sigma^{(-2,-1)}(x), \sigma^{(-2,-1)}(y) \rangle_\infty = \tau$ (right).

The characterization of Lemma 4.3 makes it possible to give an elementary proof of the non-existence of D-dimensional CA for $D \geqslant 2$. Our proof uses a pigeon-hole argument to show that, for $D \geqslant 2$, differences among configurations cannot propagate with a constant speed as required by Lemma 4.3. We include this proof since it is considerably simpler than the proof given in [19].

THEOREM 4.4. *There are no expansive D-dimensional CA for $D \geqslant 2$.*

Proof. Let $p = |\mathscr{A}|$. We prove the result for $D = 2$, but the same reasoning can be applied for all $D \geqslant 2$. For any positive integer t, we define the set $Q_t \subset \mathbf{Z}^2$ as

$$Q_t = \{\vec{v} \in \mathbf{Z}^2 \mid \|\vec{v}\|_\infty \leqslant t\}.$$

Clearly, $|Q_t| = (2t+1)^2$. Assume F is expansive, and let τ, m denote the values given by Lemma 4.3. Let z be any configuration in $\mathscr{A}^{\mathbf{Z}^2}$. For $r > \tau$ we define $B_{z,r}$ as the set of configurations which coincide with z outside Q_r. That is,

$$B_{z,r} = \{x \in \mathscr{A}^{\mathbf{Z}^2} \mid x(\vec{v}) = z(\vec{v})\ \forall \vec{v} \notin Q_r\}.$$

Clearly, $|B_{z,r}| = p^{|Q_r|} = p^{(2r+1)^2}$. Consider now any pair $x, y \in B_{z,r}$. We have $\langle x, y \rangle_\infty \leqslant r$, hence, by Lemma 4.3,

$$\exists k, \quad 0 \leqslant k \leqslant (r-\tau)\, m, \qquad \text{such that} \quad \langle F^k(x), F^k(y) \rangle_\infty \leqslant \tau. \tag{16}$$

In other words, for some $k \leqslant m$ the two configurations $F^k(x)$ and $F^k(y)$ must differ inside Q_τ. We prove that F cannot be expansive by showing that this is not possible for all pairs $x, y \in B_{z,r}$. For any configuration x, let $x(Q_\tau)$ denote the set of values assumed by x inside Q_τ. For $x \in B_{z,r}$, we define the *orbit* of x as the set

$$O_x = \bigcup_{i=0}^{(r-\tau)\, m} [F^i(x)](Q_\tau).$$

The orbit O_x represents the values assumed inside Q_τ by the sequence x, $F(x)$, $F^2(x)$, ..., $F^{(r-\tau)\, m}(x)$. If F is expansive, all orbits O_x, for $x \in B_{z,r}$, must be distinct (otherwise (16) is violated). We prove that this is impossible by showing that, for r sufficiently large, the number of possible orbits is less than $|B_{z,r}|$. The number of distinct orbits is given by $(p^{|Q_\tau|})^{(r-\tau)\, m}$ which asymptotically is $\approx p^{4\tau^2 (r-\tau)\, m}$. Since m and τ are constants, we have that for r large enough this is less than $|B_{z,r}| = p^{(2r+1)^2}$. ∎

An immediate consequence of Lemma 4.3 is that for expansive CA Lemma 4.1 holds in a much stronger sense. More precisely, we can find a lower bound to the number of iterations required to double the original distance which holds even for arbitrarily close points.

COROLLARY 4.5. *Let $(\mathscr{A}^{\mathbf{Z}}, F)$ be an expansive 1-dimensional CA. Then, there exist $\varepsilon > 0$ and an integer n such that*

$$\forall x, y \in X, \qquad 0 \neq d(x, y) \leqslant \varepsilon \;\Rightarrow\; \exists k \leqslant n\colon d(F^k(x), F^k(y)) \geqslant 2d(x, y). \quad (17)$$

Note that Lemma 4.3 holds independently of the particular metric we use. This is not true for Corollary 4.5. In fact, it takes little effort to prove that there exist metrics which induce the product topology for which Corollary 4.5 does not hold (see for example the metric proposed in [11]).

5. EXPANSIVITY AND LYAPUNOV EXPONENTS IN CELLULAR AUTOMATA

In the previous section we have shown that for any expansive CA there is a lower bound to the speed at which "perturbations" propagate (Lemma 4.3 and Corollary 4.5). A remarkable consequence of this fact is that expansive CA have positive Lyapunov exponents. In order to prove this result we need a preliminary lemma.

LEMMA 5.1. *Let $(\mathscr{A}^{\mathbf{Z}}, F)$ be an expansive 1-dimensional CA. For every $x \in \mathscr{A}^{\mathbf{Z}}$, let $\Lambda_n^+(x)$, $\Lambda_n^-(x)$ be defined as in Section 3. Then, there exists a constant $c > 0$ such that*

$$\limsup_{n \to \infty} \frac{1}{n} \Lambda_n^+(x) \geqslant c \qquad \textit{and} \qquad \limsup_{n \to \infty} \frac{1}{n} \Lambda_n^-(x) \geqslant c.$$

Proof. Since F is expansive, by Lemma 4.3 we can find τ, m such that

$$\forall y, z \in \mathscr{A}^{\mathbf{Z}} \qquad \langle y, z \rangle_\infty \geqslant \tau \Rightarrow \exists k \leqslant m\colon \langle F^k(y), F^k(z) \rangle_\infty < \langle y, z \rangle_\infty. \quad (18)$$

We prove the lemma by showing that there exists an infinite set of integers $\{n_j\}_{j>0}$ such that

$$\frac{1}{n_j} \Lambda_{n_j}^-(x) \geqslant \frac{1}{m}$$

(the proof for $\Lambda_n^+(x)$ is similar).

Let $\tilde{x}$ be a configuration such that $\tilde{x}(0) \neq x(0)$ and $\tilde{x}(i) = x(i)$ for $i \neq 0$. For any integer $j > 0$, let $y_j = \sigma^{j+\tau}(x)$ and $\tilde{y}_j = \sigma^{j+\tau}(\tilde{x})$. By construction, we have

$$\langle y_j, \tilde{y}_j \rangle_\infty = j + \tau. \tag{19}$$

By applying (18) j times we get that there exists $n_j \leqslant jm$ such that

$$\langle F^{n_j}(y_j), F^{n_j}(\tilde{y}_j) \rangle_\infty \leqslant \tau. \tag{20}$$

This means that while $y_j = \sigma^{j+s}(x)$ and $\tilde{y}_j = \sigma^{j+s}(\tilde{x})$ differ only at position $j+\tau$, $F^{n_j}(y_j)$ and $F^{n_j}(\tilde{y}_j)$ must be differ at positions $\leqslant \tau$. Since $\Lambda^-_{n_j}(x)$ measures how far a perturbation can move left in n_j steps, we have

$$\frac{1}{n_j} \Lambda^-_{n_j}(x) \geqslant \frac{j}{n_j} \geqslant \frac{1}{m}$$

as claimed. To complete the proof we must show that the set $\{n_j\}_{j>0}$ contains an infinite number of elements. To see this note that, by (19) and (20), we have

$$d(y_j, \tilde{y}_j) = 2^{-j-\tau} \qquad \text{and} \qquad d(F^{n_j}(y_j), F^{n_j}(\tilde{y}_j)) \geqslant 2^{-\tau},$$

which yields

$$d(F^{n_j}(y_j), F^{n_j}(\tilde{y}_j)) \geqslant 2^j d(y_j, \tilde{y}_j).$$

Since F is a Lipschitz function, we must have $n_j > j/\log_2 \lambda$ which proves our claim. ▮

We are now ready to prove the main result concerning Lyapunov exponents.

THEOREM 5.2. *let* $(\mathscr{A}^{\mathbf{Z}}, F)$ *be an expansive* 1-*dimensional CA, and let* Y *denote the subset of* $\mathscr{A}^{\mathbf{Z}}$ *for which the right and left Lyapunov exponents* (λ^+ *and* λ^-) *exist. Then, there exists a constant* $c > 0$ *such that for all* $x \in Y$,

$$\lambda^+(x) \geqslant c \qquad \textit{and} \qquad \lambda^-(x) \geqslant c.$$

Moreover, for any σ *invariant and* F-*invariant measure* μ *there exists a* μ-*measurable set* Z_μ *such that* $Z_\mu \subseteq Y$ *and* $\mu(Z_\mu) = 1$.

Proof. Since $\lambda^+(x)$ and $\lambda^-(x)$ are defined as

$$\lambda^+(x) = \lim_{n\to\infty} \frac{1}{n} \Lambda_n^+(x), \qquad \lambda^-(x) = \lim_{n\to\infty} \frac{1}{n} \Lambda_n^-(x)$$

if the limits exist they cannot be smaller than the constant c given by Lemma 5.1. The second part of the theorem follow directly by Theorem 3.1. ∎

Note that, if F is expansive it is also surjective (see for example [9]). By a result in [6] we know that the Haar measure is F-invariant and σ-invariant and therefore satisfies the hypothesis of Theorem 5.2.

6. EXPANSIVITY OVER INVARIANT SUBSPACES

In this section we prove that there exist non-expansive D-dimensional CA which are expansive on a subset Y of the whole phase space. Moreover, the subset Y can be chosen to be a *dense* subset of the whole space. For $D \geqslant 2$ this result is particularly significant since it shows that expansivity can be achieved if we restrict our attention to suitable subset of the configuration space $\mathscr{A}^{\mathbf{Z}^D}$.

A similar analysis, applied to different dynamical properties (topological transitivity and sensitivity to initial conditions), has been carried out by Knudsen [14] in the general framework of continuous transformations of bounded metric spaces. He proved the following result.

THEOREM 6.1 [14]. *Let $F: X \to X$, be a continuous transformation of a bounded metric space (X, d). Let Y be a dense subset of X such that $F(Y) \subseteq Y$. Then F is topologically transitive [resp. sensitive to initial conditions] over Y iff it is topologically transitive [resp. sensitive to initial conditions] over X.*

Given a CA $(\mathscr{A}^{\mathbf{Z}^D}, F)$ we say that $X \subseteq \mathscr{A}^{\mathbf{Z}^D}$ is an *invariant* subspace iff $F(X) \subseteq X$. If X is invariant, we say that (X, F) is a *subsystem* of $(\mathscr{A}^{\mathbf{Z}^D}, F)$. If X is also a dense subset of $\mathscr{A}^{\mathbf{Z}^D}$ we say that (X, F) is a *dense subsystem* of $(\mathscr{A}^{\mathbf{Z}^D}, F)$. Note that X is a dense subset of $\mathscr{A}^{\mathbf{Z}^D}$ iff for every $x \in \mathscr{A}^{\mathbf{Z}^D}$ and $k > 0$ there exists $x' \in X$ such that $\langle x, x' \rangle_\infty > k$.

Theorem 6.1 guarantees that if a map F is transitive [sensitive] over a particular dense invariant subset of the phase space, then F is transitive [sensitive] over any other dense invariant subset and on the whole space. The results of this section show that expansive maps have a different behavior.

THEOREM 6.2. *There exists a CA* $(\{0,1\}^{\mathbf{Z}}, F)$ *which is not expansive but it has an expansive dense subsystem* (X, F).

Proof. Let $F: \{0,1\}^{\mathbf{Z}} \to \{0,1\}^{\mathbf{Z}}$ be defined by

$$[F(x)](i) = (x(i) + x(i+1)) \bmod 2 \qquad x \in \{0,1\}^{\mathbf{Z}}, \quad i \in \mathbf{Z}. \tag{21}$$

F is a linear map and it is not expansive by Theorem 7 in [16]. Let

$$X = \{x \in \{0,1\}^{\mathbf{Z}}: \sigma^k(x) = x \text{ for some } k > 0\}$$

denote the set of spatially periodic configurations. One can easily verify that (X, F) is a dense subsystem of $(\{0,1\}^{\mathbf{Z}}, F)$. We prove the theorem by showing that F is expansive over X.

Given any pair of distinct configurations $x, y \in X$, let $k, h > 0$ be such that $x = \sigma^k(x)$ and $y = \sigma^h(y)$. Then, for every $m \in \mathbf{Z}$,

$$x(i) \neq y(i) \quad\Rightarrow\quad x(i+mkh) \neq y(i+mkh). \tag{22}$$

For $x, y \in X$, define

$$\langle x, y\rangle_\infty^+ = \min\{i \geqslant 0: x(i) \neq y(i)\}, \qquad \langle x, y\rangle_\infty^- = \max\{i < 0: x(i) \neq y(i)\}.$$

The symbol $\langle x, y\rangle_\infty^+$ [$\langle x, y\rangle_\infty^-$] denotes the smallest non-negative [the largest negative] position in which x and y differ. Both $\langle x, y\rangle_\infty^+$ and $\langle x, y\rangle_\infty^-$ are well defined in view of (22). By (21) we have

$$\langle x, y\rangle_\infty^- < \langle x, y\rangle_\infty^+ - 1 \quad\Rightarrow\quad \begin{cases} \langle F(x), F(y)\rangle_\infty^- \leqslant \langle x, y\rangle_\infty^- \\ \langle F(x), F(y)\rangle_\infty^+ = \langle x, y\rangle_\infty^+ - 1 \end{cases} \tag{23}$$

We prove that F is expansive over X with parameter $\delta = 1/2$. Let $x, y \in X$ be such that $d(x, y) \leqslant 1/8$; by (2) we have

$$\langle x, y\rangle_\infty^- \leqslant -3 \qquad \text{and} \qquad \langle x, y\rangle_\infty^+ \geqslant 3.$$

By (23) we get that after $k = \langle x, y\rangle_\infty^+ - 1$ iterations $\langle F^k(x), F^k(y)\rangle_\infty^+ = 1$, which implies $d(F^k(x), F^k(y)) > 1/2$.

This completes the proof. ∎

Note that a result similar to Corollary 4.5 cannot hold if a map F is expansive only over a dense subspace $X \subset \mathscr{A}^{\mathbf{Z}}$. That is, the number of iterations required to double the distance between two configurations cannot be uniformly bounded. Analogously, a result similar to Lemma 4.3

does not hold for expansive subsystems. For example, for the subsystem given in Theorem 6.2 it easy to see that for every $t \geqslant 1$ and $n > 0$ there exist $x_n, y_n \in X$ such that $\langle x_n, y_n \rangle_\infty = t$ and $\langle F^i(x_n), F^i(y_n) \rangle_\infty \geqslant t$ for $i = 1, ..., n$.

We now show that also in the 2-dimensional case there exist (non-expansive) CA which are expansive over a dense subset of the whole space. Note that the construction given below can be easily generalized to show that the same result holds true also for $D > 2$.

THEOREM 6.3. *There exists a 2-dimensional CA* $(\{0, 1\}^{\mathbf{Z}^2}, F)$ *which has an expansive dense subsystem* (Y, F).

Proof. Let $F \equiv \sigma^{(-1, -1)}$. The map F is simply a one-step shift in the direction of the main diagonal of the lattice. The construction of the subset Y over which F is expansive is quite complex. The basic idea is to force any pair of configurations $x, y \in Y$ to differ on infinitely many positions situated along the main diagonal of the lattice. The iteration of the map F will move one of these differences to position $(0, 0)$ so that, by (2), for some $k > 0$ we have $d(F^k(x), F^k(y)) = 1$.

Let X denote the set of configurations $x \in \{0, 1\}^{\mathbf{Z}^2}$ such that: (a) there exists only a finite number of pairs (i, j) such that $i \neq j$ and $x(i, j) \neq 0$, and (b) there exists $k \in \mathbf{Z}$ such that for every $i > k$ we have $x(i, i) = 0$. We split any configuration $x \in X$ into two parts: the *body* and the *tail*. The body B_x of x is the smallest *square region* of the lattice which satisfies the following properties:

1. the cells on the diagonal of B_x have coordinates (i, i), i.e., the diagonal of B_x lies on the main diagonal of the lattice;

2. each cell not in B_x which contains 1 lies on the main diagonal of the lattice, that is, B_x contains all off-diagonal 1's;

3. let (u, u) denote the coordinates of the upper right corner of B_x; for every $i > u$ we have $x(i, i) = 0$.

Let (l, l) denote the coordinates of the lower left corner of B_x. The tail T_x of x is defined by

$$T_x = \{x(l-1, l-1), x(l-2, l-2), x(l-3, l-3), ...\}.$$

Note that, by construction, all nonzero cells of x belongs to $B_x \cup T_x$. Let C be the map which associates to each body B_x the binary sequence obtained by reading the entries of B_x row by row, left to right, from top to bottom. Let E: $\{0, 1\}^* \to \mathbf{N}$ be any injective map, where $\{0, 1\}^*$ denotes

$$
x = \begin{array}{cccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & & & & & & & & & \cdots \\
\cdots & & & & & 0 & 1 & 0 & & \cdots \\
\cdots & & & & & 1 & 1 & 0 & & \cdots \\
\cdots & & & & & 0 & 0 & 0 & & \cdots \\
\cdots & & & & 1^2 & & & & & \cdots \\
\cdots & & & 0 & & & & & & \cdots \\
\cdots & & 1^4 & & & & & & & \cdots \\
\cdots & 0 & & & & & & & & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

$$C(B_x) = \{0,1,0,1,1,0,0,0,0\},\ E(C(B_x)) = 2,\ T_x = \{1,1,0,1,1,1,1,0,1^6,\ldots\}.$$

FIG. 2. Example of admissible configuration. B_x and T_x denote the body and the tail of x, respectively. In this example we assume that $E(C(B_x)) = 2$. All entries outside B_x and T_x are zero.

the set of binary sequences of finite length. We say that a configuration $x \in X$ is admissible (see Fig. 2) iff

$$T_x = \{1^{E(C(B_x))}, 0, 1^{2E(C(B_x))}, 0, 1^{3E(C(B_x))}, \ldots\},$$

where 1^n denotes the sequence of n consecutive 1's.

Let Y denote the set of admissible configurations. It takes little effort to prove that Y is a dense subset of $\{0, 1\}^{\mathbf{Z}}$ (the content and the size of the body can be arbitrarily chosen), and that $F(Y) \subseteq Y$ (the main diagonal of the lattice is mapped by F over itself). To see that F is expansive over Y consider any pair of distinct configurations $x, y \in Y$. One of the following two cases must hold:

$$\text{(a)}\quad C(B_x) \neq C(B_y), \qquad \text{(b)}\quad C(B_x) = C(B_y) \quad \text{and} \quad x = F^n(y).$$

In case (a), since E is injective, $E(C(B_x)) \neq E(C(B_y))$. Hence the tails T_x and T_y must differ at an infinite number of diagonal positions. In case (b), since tails, by construction, are *aperiodic* sequences, we conclude again that x and y must differ at an infinite number of diagonal positions. In both cases, after a finite number k of iterations we have $[F^k(x)](0,0) \neq [F^k(y)](0,0)$ which implies $d(F^k(x), F^k(y)) = 1$.

This completes the proof. ∎

The subspace Y defined in the proof of Theorem 6.3 is a countable set. We now show how to modify the construction of Y in order to get an uncountable subspace Y' such that (Y', F) is still an expansive dense subsystem.

We use the notation introduced in the proof of Theorem 6.3. Let $x \in X$ and $n = E(C(B_x))$. Now we say that x is an admissible configuration if

$$T_x = \{1^n, 0, a_1, 0, 1^{2n}, 0, a_2, 0, 1^{3n}, ..., 1^{in}, 0, a_i, 0, 1^{(i+1)n}, ...\},$$

where the binary sequence $\{a_j\}_{j \in N}$ is any sequence which satisfies the following conditions:

1. $a_q = 0$, if q is not a prime power;
2. $a_{p^n} = a_{p^m}$, for every prime p and every pair of positive integers n, m.

Let Y' denote the new set of admissible configurations. Note that, since there are infinitely many primes, there are uncountable many sequences $\{a_j\}_{j \in N}$ which can appear in the tail of admissible configurations. Hence, the set Y' is uncountable. With some additional work with respect to the proof of Theorem 6.3 it is possible to prove that any pair of distinct configurations $x, y \in Y'$ must differ on infinitely many diagonal positions. This implies that (Y', F) is expansive subsystem as claimed.

7. EXPANSIVITY VERSUS SENSITIVITY IN CELLULAR AUTOMATA

In this section we consider sensitive Lipschitz functions over compact metric spaces. Since the definitions of sensitivity and expansivity are similar, it is natural to ask whether results analogous to Lemma 4.1 and Corollary 4.5 hold true also for sensitive CA. As we will see, this is not the case since sensitive and expansive maps turn out to have a quite different behavior.

The following example shows a sensitive 1-dimensional Ca $(\mathscr{A}^{\mathbf{Z}}, F)$ such that

$$\begin{aligned} \forall n \quad \exists \varepsilon_n, x_n : d(x_n, y) \leqslant \varepsilon_n \\ \Rightarrow \quad d(F^k(x_n), F^k(y)) \leqslant 2d(x_n, y) \qquad \text{for} \quad 0 \leqslant k \leqslant n. \end{aligned} \tag{24}$$

In other words, there is no upper bound to the number of iterations required to double the distance between two arbitrarily close configurations or, equivalently, there is no upper bound to the number of iterations required to move a perturbation front of at least one position.

```
···*****111110000000000000000*****···
···*****111111000000000000000*****···
···*****111111100000000000000*****···
···*****111111110000000000000*****···
···*****111111111000000000000*****···
···*****111111111100000000000*****···
```

FIG. 3. The evolution of configuration y. Since $f(\alpha, 1, 1)=1$, and $f(0, 0, \alpha)=0$ the positions marked with $*$ do not affect the central section until there are 0's left.

EXAMPLE 3. We consider the 1-dimensional CA over the alphabet $\mathscr{A}=\{0, 1, 2\}$ defined by the following local rule (here α, β denote any symbol in $\mathscr{A}$):

$$f(\alpha, 1, 2)=2, \qquad f(\alpha, 1, 1)=1, \qquad f(\alpha, 1, 0)=1, \tag{25}$$

$$f(0, 0, \alpha)=0, \qquad f(1, 0, \alpha)=1, \qquad f(2, 0, \alpha)=1, \tag{26}$$

$$f(\alpha, 2, \beta)=1. \tag{27}$$

For $n>0$ we define x_n, ε_n as follows. We set $m=\lceil n/2\rceil$, $\varepsilon_n=2^{-m-1}$ and

$$x_n(i)=\begin{cases}0, & \text{if } \ i>-m,\\ 1, & \text{if } \ i\leqslant -m.\end{cases}$$

Given y such that $d(x_n, y_n)\leqslant\varepsilon_n$, let t_1, t_2 be such that

$$x_n(-t_1)\neq y(-t_1), \qquad x_n(t_2)\neq y(t_2), \qquad x_n(i)=y(i) \quad \text{for} \quad -t_1<i<t_2.$$

In the following we assume that both t_1 and t_2 are finite but the same reasoning holds also if one of them is not finite. Note that our choice of ε_n implies that both t_1 and t_2 are greater than m. The fundamental observation is that the values $y(i)$ with $i\geqslant t_2$ or $i\leqslant -t_1$ do not affect the values $[F^k(y)](i)$ for $k\leqslant n$ and $-t_1<i<t_2$ (see Fig. 3). Hence, for $k\leqslant n$,

$$d(F^k(x_n), F^k(y))\leqslant \max\left\{\frac{1}{2^{t_2}}, \frac{1}{2^{t_1}}\right\}\leqslant d(x_n, y).$$

which proves (24).

To prove that F is sensitive (with constant 1/2), we show that $\forall x\in\mathscr{A}^{\mathbf{Z}}$ and $\forall\varepsilon$ there exist y, z and an integer n such that

$$d(x, y)\leqslant\varepsilon, \qquad d(x, z)\leqslant\varepsilon, \qquad d(F^n(y), F^n(z))\geqslant 1. \tag{28}$$

$$x$$

$$\cdots 220110122\boxed{010011101}001112122\cdots$$

$\Downarrow$	$\Downarrow$
y	z
$\cdots 111111111\boxed{010011101}111111111\cdots$	$\cdots 111111111\boxed{010011101}111111112\cdots$
$\downarrow$	$\downarrow$
$\cdots 111111111\boxed{111111111}111111111\cdots$	$\cdots 111111111\boxed{111111112}111111111\cdots$
$\downarrow$	$\downarrow$
$\cdots 111111111\boxed{111111111}111111111\cdots$	$\cdots 111111111\boxed{111121111}111111111\cdots$

FIG. 4. Sample configurations x, y and z for $m=4$. We show y, $F^9(y)$, $F^{13}(y)$ (left) and z, $F^9(z)$, $F^{13}(z)$ (right).

Given $0<\varepsilon<1/2$, we choose m such that $x(i)=y(i)$ for $|i|\leqslant m$ implies $d(x, y)<\varepsilon$. We define y and z as follows (see also Fig. 4):

$$y(i)=\begin{cases} x(i), & \text{if } |i|\leqslant m;\\ 1, & \text{if } |i|>m;\end{cases} \qquad z(i)=\begin{cases} y(i), & \text{if } i\neq 3m+1;\\ 2, & \text{if } i=3m+1.\end{cases}$$

To prove (28) we show that $[F^{3m+1}(y)](0)=1$ and $[F^{3m+1}(z)](0)=2$. We first consider the simpler case in which $x(i)$: $\neq 2$ for $|i|\leqslant m$ (see Fig. 4). After $2m+1$ steps we have $[F^{2m+1}(y)](i)=1$ for all i. In fact, since y contains no 2's, by (25) $y(i)=1 \Rightarrow [F^k(y)](i)=1$ for all $k>0$. Vice versa, since $f(1, 0, \alpha)=1$, the number of 0's decreases at each step until none is left. Similarly, we have $[F^{2m+1}(z)](i)=1$ for $i\neq m$, and $[F^{2m+1}(z)](m)=2$ (since $f(\alpha, 1, 2)=2$, the value 2 initially in position $3m+1$ moves left by one position at each step). After m more steps, we have $[F^{3m+1}(y)](0)=1$, and $[F^{3m+1}(z)](0)=2$ as claimed.

Now consider a generic $x\in\mathscr{A}^{\mathbf{Z}}$. By (26) we have that the number of 0's decreases by at least one at each step until none is left. Since $(\alpha, 1, 2)$ is the only triplet that generates a 2, we have that a symbol 2 "survives" only by moving left. Hence after $2m+1$ steps we have $[F^{2m+i}(y)](i)=1$ for $i\geqslant -m$. Similarly, $[F^{2m+1}(z)](i)=1$ for $i\geqslant -m$ $i\neq m$, and $[F^{2m+1}(z)](m)=2$. After m more steps, we have $[F^{3m+1}(y)](0)=1$ and $[F^{3m+1}(z)](0)=2$ which proves (28). ∎

8. FURTHER WORK

Deciding if a given dynamical system satisfies a certain property is one of the most important problem in the theory of DTDS, but few results are known even for the special case of CA. Amoroso and Patt [1] showed that surjectivity and injectivity of 1-dimensional CA are decidable properties, while Kari [12] proved that the same properties are undecidable in any dimension greater than 1. The decidability of expansivity and other topological properties (such as sensitivity, transitivity, denseness of periodic orbits, *etc*.) are challenging open problems.

Our results suggest that expansivity of CA could be a decidable property. The crucial observation is that Lemma 4.3 guarantees that an expansive behavior must *become apparent*, i.e., algorithmically detectable, after a bounded number of iterations. Indeed, using Lemma 4.3 and working out some hairy details, it is possible to prove that expansivity is at least semidecidable, i.e., there exists an algorithm which, in finite time, answers yes if it receives as input an expansive CA. We are currently investigating algorithms for testing non-expansivity.

ACKNOWLEDGMENT

We thank A. Maass for the many valuable comments on an early version of the manuscript.

REFERENCES

1. Amoroso, S., and Patt, Y. (1972), Decision procedures for surjectivity and injectivity of parallel maps for tessellation structures, *J. Comput. System Sci.* **6**, 448–464.
2. Assaf, D., IV, and Coppel, W. A. (1992), Definition of chaos, *Amer. Math. Monthly* **99**, 865.
3. Banks, J., Brooks, J., Cairns, G., Davis, G., and Stacey, P., (1992), On the Devaney's definition of chaos, *Amer. Math. Monthly* **99**, 332–334.
4. Blanchard, F., Kurka, P., and Maass, A. (1997), Topological and measure-theoretic properties of one-dimensional cellular automata, *Physica D* **103**, 86–99.
5. Cattaneo, G., Formenti, G., Margara, L., and Mazoyer, J. (1997), A shift-invariant metric on S^{Z} inducing a non-trivial topology, *in* "22nd International Symposium on Mathematical Foundations of Computer Science (MFCS'97)," Springer-Verlag, Berlin/New York.
6. Coven, E., and Paul, M. (1974), Endomorphisms of irreducible subshifts of finite type, *Math. Systems Theory* **8**, 167–175.
7. Denker, M., Grillenberger, C., and Sigmund, K. (1976), "Ergodic Theory on Compact Spaces," Lecture Notes in Mathematics, Vol. 127, Springer-Verlag, Berlin/New York.
8. Devaney, R. L. (1989), "An Introduction to Chaotic Dynamical Systems," Addison–Wesley, Reading, MA.
9. Fagnani, F., and Margara, L. Expansivity, permutivity, and chaos for cellular automata, *Theory Comput. Systems*, to appear.
10. Garzon, M. (1995), "Models of Massive Parallelism," EATCS Texts in Theoretical Computer Science, Springer-Verlag, Berlin/New York.
11. Hedlund, G. A. (1969), Endomorphism and automorphism of the shift dynamical system, *Math. Systems Theory* **3**, 320–375.

12. Kari, J. (1994), Reversibility and surjectivity problems of cellular automata, *J. Comput. System Sci.* **48**, 149–182.

13. Knudsen, C. (1994), "Aspects on Noninvertible Dynamics and Chaos," Ph.D. thesis Physics Department, Technical University of Denmark.

14. Knudsen, C. (1994), Chaos without nonperiodicity, *Amer. Math. Monthly* **101**, 563–565.

15. Kurka, P. (1997), Equicontinuity and attractors in cellular automata, *Ergodic Theory Dynam. Systems* **17**, 417–433.

16. Manzini, G., and Margara, L. (1997), A complete and efficiently computable topological classification of D-dimensional linear cellular automata over $\mathbf{Z}_m$, *in* "24th International Colloquium on Automata Languages and Programming (ICALP'97)," Springer-Verlag, Berlin/New York.

17. Martelli, M. (1994), "Discrete Dynamical Systems and Chaos," Pitman Monographs and Surveys in Pure and Applied Science, Vol. 62, Longman, Harlow/New York.

18. Shereshevsky, M. A. (1992), Lyapunov exponents for one-dimensional cellular automata, *J. Nonlinear Sci.* **2**, 1–8.

19. Shereshevsky, M. A. (1993), Expansiveness, entropy and polynomial growth for groups acting on subshifts by automorphisms, *Indag. Math.* (*N.S*) **4**, 203–210.

20. Vellekoop, M., and Berglund, R. (1994), On intervals, transitivity = chaos, *Amer. Math. Monthly* **101**, 353–355.